

A Plasma Analogy and Berry Matrices for Non-Abelian Quantum Hall States

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We present an approach to the computation of the non-Abelian statistics of quasiholes in quantum Hall states, such as the Pfaffian state, whose wavefunctions are related to the conformal blocks of minimal model conformal field theories. We use the Coulomb gas construction of these conformal field theories to formulate a plasma analogy for the quantum Hall states. A number of properties of the Pfaffian state follow immediately, including the Berry phases which demonstrate the quasiholes' fractional charge, the abelian statistics of the two-quasihole state, and equal-time ground state correlation functions. The non-Abelian statistics of multi-quasihole states follows from an additional assumption.

Introduction. Recent investigations of the possibility of finding non-Abelian braiding statistics in the quantum Hall regime have used a remarkable observation due to Moore and Read as a springboard [1–5]. They drew attention to the fact that certain wavefunctions which have been proposed to describe incompressible quantum Hall states could be represented as conformal blocks in corresponding conformal field theories. This is not a statement about dynamics in the quantum Hall regime¹, but merely a statement that some wavefunctions could be reconstructed from conformal field theories. Hence, it might not appear particularly useful. However, this observation suggests a way around a technical impasse which impedes the study of non-Abelian quantum Hall states. The problem and a strategy for its circumvention noted by Moore and Read may be summarized as follows. Suppose we initially eschew any attempt to realistically account for the physics of the long-range Coulomb interaction, finite thickness of the quantum well, inter-Landau-level mixing, and other complications, and, instead, work with an unrealistic but tractable model Hamiltonian whose exact zero-energy eigenstates (ground state and multi-quasihole states) can be explicitly constructed. The statistics of these quasiholes may be determined from the exact eigenstates using the Berry phase technique [6]. If the model Hamiltonian may be adiabatically connected to the one which governs the actual physical 2D electron gas – i.e. if the two Hamiltonians are in the same universality class – the statistics of the quasiholes and other ‘topological quantum numbers’ [7] will be the same for both. In such a case, the calculation of these properties of the idealized Hamiltonian will be relevant to the real system. The stumbling block is that Berry phases (or Berry matrices in the non-Abelian case) are determined from matrix elements of the eigenstates and further progress is impossible unless we can find a method for calculating the matrix elements of the correlated many-body wavefunctions of interest.

In his pioneering work on the fractional quantum Hall effect [8], Laughlin took advantage of the equality between the squared modulus of his wavefunctions and the Boltzmann weight of a classical two-dimensional plasma. The conventional wisdom on plasmas implies that the Laughlin states are quantum liquids [8] and has also facilitated calculations of off-diagonal long-range-order in these states [7]. The original calculation of the fractional statistics of the Laughlin quasiholes [6] relied on the incompressible liquid nature of the states but did not explicitly utilize the plasma analogy. However, the calculation may be rephrased [2] in the following way so that its dependence on the plasma analogy is emphasized. Let us write the wavefunction for a Laughlin state with two quasiholes at η_1 and η_2 in the following form (ℓ_0 is the magnetic length):

$$\Psi_{\eta_1 \eta_2} = (\eta_1 - \eta_2)^{1/m} \prod_k (z_k - \eta_1) (z_k - \eta_2) \prod_{i>j} (z_i - z_j)^m e^{-\frac{1}{4\ell_0^2} \frac{1}{m} (|\eta_1|^2 + |\eta_2|^2)} e^{-\frac{1}{4\ell_0^2} \sum |z_i|^2}. \quad (1)$$

Ordinarily, we would choose the wavefunction to be a single-valued function of the quasihole coordinates (which are simply some parameters in an electron wavefunction) by, for instance, replacing the first factor by its modulus or

¹At least on the face of it. In fact, in many cases the conformal field theory which reproduces the wavefunctions is the same as the one which describes the gapless excitations at the edge. If the effective field theory of the bulk state is a Chern-Simons theory, this relationship can be understood in light of Witten’s discoveries [10]: the states of the Chern-Simons theory which describes the bulk are isomorphic to the conformal blocks of the CFT which reproduces the bulk wavefunction. At the same time, the boundary theory which is concomitant with the Chern-Simons theory is precisely the CFT of the edge. The equality of these CFTs follows from the general covariance of the Chern-Simons theory.

removing it altogether. In (1), we have intentionally chosen the phase of the wavefunction such that it is a multi-valued function of the quasihole coordinates (but, of course, single-valued in the electron coordinates); the net effect of braiding the two quasiholes will be the Berry holonomy together with the explicit monodromy of (1). Only this combination is physically observable, and by changing the phase of the wavefunction – i.e. performing a gauge transformation – we can divide this phase between the Berry holonomy and the explicit monodromy in any way we wish. As we will see momentarily, the choice (1) is particularly convenient because it is the ‘fractional statistics gauge’ [9] in which the Berry holonomy vanishes and the explicit monodromy is the whole story. A second advantage of (1) is that it is a holomorphic function of η_1, η_2 up to the Gaussian factor. Hence, the Berry holonomy, γ , acquired when η_1 is taken adiabatically around η_2 is:

$$\gamma = \oint d\eta_1 \langle \Psi_{\eta_1 \eta_2} | \frac{\partial}{\partial \eta_1} | \Psi_{\eta_1 \eta_2} \rangle = \oint d\eta_1 \frac{\partial}{\partial \eta_1} \langle \Psi_{\eta_1 \eta_2} | \Psi_{\eta_1 \eta_2} \rangle + \frac{\pi}{m} \frac{\Phi}{\Phi_0} \quad (2)$$

where Φ is the magnetic flux enclosed within the path taken by η_1 and Φ_0 is the flux quantum. The second equal sign follows from the fact that $\langle \Psi_{\eta_1 \eta_2} |$ depends only on $\bar{\eta}_1$ and not on η_1 , except through the Gaussian factor, which yields the second term on the right hand side. This term represents the fractional charge of the quasiparticles and does not depend on whether or not η_1 encircles η_2 . We now argue that as a result of the prefactor chosen in (1), the first term vanishes. This follows from the observation that this inner product is the partition function of a classical plasma of charge \sqrt{m} particles at temperature $T = 1$ with a neutralizing background of density $1/2\pi\sqrt{m}$ and two fixed charge $1/\sqrt{m}$ particles at η_1 and η_2 :

$$\langle \Psi_{\eta_1 \eta_2} | \Psi_{\eta_1 \eta_2} \rangle = \int \prod_i d^2 z_i e^{\frac{2}{m} \ln |\eta_1 - \eta_2| + \sum_{i, \alpha} 2 \ln |\eta_\alpha - z_j| + \sum_{i > j} 2m \ln |z_i - z_j| - \frac{1}{m} \frac{1}{4\epsilon_0^2} (|\eta_1|^2 + |\eta_2|^2) - \frac{1}{4\epsilon_0^2} \sum |z_i|^2} \quad (3)$$

(Usually, one takes $T = m$ and has charge m electrons and charge 1 quasiholes, but the above convention is more natural for what follows.) When $|\eta_1 - \eta_2|$ is larger than the Debye screening length, the interaction between these two charges will be screened, so the partition function will be independent of η_1, η_2 . In other words, $\langle \Psi_{\eta_1 \eta_2} | \Psi_{\eta_1 \eta_2} \rangle$ is independent of η_1 and η_2 so long as they are well-separated. Therefore, the Berry holonomy vanishes and the fractional statistics is given by the explicit monodromy of (1).

Unfortunately, this argument appears limited to the Laughlin states; the squared moduli of the states which are promising hunting grounds for non-Abelian statistics, such as the Pfaffian state,

$$\Psi_{\text{Pf}} = \text{Pf} \left(\frac{1}{z_i - z_j} \right) \prod_{i > j} (z_i - z_j)^2 e^{-\frac{1}{4\epsilon_0^2} \sum |z_i|^2} \quad (4)$$

(where $\text{Pf} \left(\frac{1}{z_i - z_j} \right) = \mathcal{A} \left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \dots \right)$ is the antisymmetrized product over pairs of electrons) do not seem to be equal to the Boltzmann weight of a classical plasma, at least not in the most obvious way. A further complication arises from the fact that there is a degenerate set of multi-quasihole states in the putative non-Abelian cases, so off-diagonal matrix elements must also be calculated in order to obtain a Berry matrix. It is at this seemingly hopeless point that Moore and Read’s observation can come to the rescue. Recall that the conformal blocks of certain conformal field theories are related to many of the quantum Hall states – including the Laughlin states and the Pfaffian state – which are exact ground states of model Hamiltonians. It is natural to conjecture that the quasihole statistics are equal to the monodromy properties of the conformal blocks (see also [2]). These statistics are encapsulated in the concomitant Chern-Simons theories [10] which describe the long-wavelength physics of the quantum Hall state [7]. Some justification for this conjecture, on rather general grounds, was given in [4], where it was used to conclude that the braiding matrices of the $2n$ -quasihole states in the Pfaffian are embedded in the spinor representation of $SO(2n)$.

In this paper, we give further arguments (which we believe can serve as an outline for a proof) supporting this conjecture. We formulate a plasma analogy for non-Abelian quantum Hall states which makes use of the fortunate fact that all of the minimal model conformal field theories have a Coulomb gas description. At a first glance, this seems tantalizing but insufficient for our purposes because the construction involves complicated contour integrals of the locations of screening charges. However, following Mathur [11], we rewrite these contour integrals as two-dimensional integrals. As a result, we find that sums of the desired inner products can be expressed as the partition functions of plasmas with an additional large number of screening charges (which only enhance the screening property which we invoke). The individual inner products are obtained by analytic continuation. This approach is applied to the Pfaffian state, which is associated with the $c = \frac{1}{2} + 1$ conformal field theory. Our plasma analogy enables us to compute the electron density matrix, the charge of the quasihole excitations (which, we argue, have localized charge distributions), and the Abelian statistics of the two-quasihole state. With one assumption regarding the analytic continuation, we

can show that the Berry matrices of states with an arbitrary number of quasiholes are trivial and the non-Abelian statistics can be obtained entirely from the monodromy properties of the conformal blocks, in agreement with [4].

Background Charge Construction and a Plasma Analogy. Let us momentarily take one step back before taking two steps forward. Recall that the Laughlin state can be expressed as a conformal block of the $c = 1$ theory, which is the theory of a free bosonic field φ :

$$S = \frac{1}{2\pi} \int \partial\varphi \bar{\partial}\varphi \quad (5)$$

The Laughlin state is given by the conformal block of the vertex operator $V_{\sqrt{m}}(z) =: \exp(i\sqrt{m}\varphi_R(z))$:, (φ_R is the holomorphic part of the boson field) which plays the role of the electron, together with the operator for a neutralizing background charge.

$$\Psi_{1/m} = \langle e^{i\sqrt{m}\phi(z_1)} e^{i\sqrt{m}\phi(z_2)} \dots e^{i\sqrt{m}\phi(z_N)} e^{-i \int d^2z \sqrt{m}\rho_0\phi(z)} \rangle \quad (6)$$

Conformal blocks with $V_{1/\sqrt{m}}(\eta_i) =: \exp(i\varphi_R(\eta_i)/\sqrt{m})$: are equal to states with quasiholes at η_i . The product of the Laughlin wave function and its complex conjugate can be written as the product of a holomorphic and an antiholomorphic conformal block or, simply, as a correlation function of the left-right symmetric field $V_{\sqrt{m}}(z, \bar{z}) =: \exp(i\sqrt{m}(\varphi_R(z) + \varphi_L(\bar{z})))$:. These correlation functions admit the Coulomb gas or plasma description of (3). The Pfaffian state, however, involves in its description a $c = \frac{1}{2}$ conformal field theory. Can we describe it, too, by means of a Coulomb gas?

Naively, the answer is yes since any conformal field theory admits a Coulomb gas description, as was discovered by Feigin and Fuchs and Dotsenko and Fateev (see [12] and references therein). The Dotsenko-Fateev picture involves putting a certain *background charge* at infinity and introducing *screening operators* in a $c = 1$ conformal block. Then any conformal block of any conformal field theory can be expressed as

$$F(z_1, z_2, \dots) = \int_{\mathcal{C}_i} \prod_i dw_i \left\langle \prod_j O_j(w_j) \prod_k V_k(z_k) \right\rangle \quad (7)$$

where the vertex operators $O(w_j)$ are the screening operators and $V_k(z_k)$ are the vertex operators of interest. The integration of the screening operator locations is along certain contours, \mathcal{C}_i . The choice of contours determines which conformal blocks we obtain. Notice that the block on the right hand side of (7) is a standard Coulomb gas. Therefore it's tempting to conclude that we have a plasma analogy for any wave function connected to conformal field theory.

However, this conclusion is premature. The contours for the integrals of (7) are in general hard to define. To compute inner products we would have to introduce holomorphic and antiholomorphic screening charges, each integrated over its own contour. Since, in general, the number of screening charges we need will be of the order of the number of the electrons, it is unclear how we can compute the desired inner products. It is also unclear if we are allowed to exchange the order of integration over the positions z_i of the electrons (which we will eventually have to do in order to compute the inner products) with the contour integrations over the screening charge locations since these contours are supposed to wind around the electron positions in a complicated way [12].

A way out of this quagmire can be found in a paper by Mathur [11]. He suggested the following. If, instead of considering holomorphic vertex operators, we consider full vertex operators which are the products of the holomorphic and antiholomorphic ones, $V(z, \bar{z}) \equiv V(z)V(\bar{z})$, then it can be shown that

$$\int \prod_i d^2w_i \left\langle \prod_j O(w_j, \bar{w}_j) \prod_k V(z_k, \bar{z}_k) \right\rangle = \sum_n C_n F_n(z_1, z_2, \dots) F_n(\bar{z}_1, \bar{z}_2, \dots) \quad (8)$$

where the sum over n is the sum over different conformal blocks F_n (different choices of contours in (7)) while the C_n are the so-called *structure coefficients*, which make the combination on the right-hand-side of (8) a single valued function.

The advantage of (8) over (7) is that the integration of the screening charges is over the entire complex plane, thereby putting the screening charges and electrons on equal footing in the computation of inner products. The disadvantage of (8) is that instead of producing the squared modulus of a particular conformal block (corresponding to a particular wave function), it gives us a linear combination of them. We will discuss ways of getting around this problem later. For now, let us concentrate on cases in which there is only one conformal block, making the summation over n in (8) unnecessary.

As an example, we consider the Pfaffian state (4), which is a conformal block of the $c = \frac{1}{2} + 1$ theory [1,4]. The Pfaffian factor is the $c = \frac{1}{2}$ part, while the rest comes from the $c = 1$ part and has the standard plasma form of a

Laughlin state. The Laughlin plasma consists of charge $\sqrt{2}$ electrons. According to [12], the $c = \frac{1}{2}$ part can also be described as a plasma with N charges of magnitude $3/\sqrt{6}$ or $-2/\sqrt{6}$ associated with the electrons together with $\approx \frac{N}{2}$ screening charges of magnitude $4/\sqrt{6}$ or $-3/\sqrt{6}$ which also roam the entire complex plan and the background charge $-1/\sqrt{6}$ which has to be taken to the infinitely far away point. In other words, the background charge, being at infinity, does not interact with other charges, but shifts the total charge balance. The sum of all the other charges of our plasma has to be equal to minus the background charge (details can be found in [12]). We note that at large N it does not affect the neutrality of the plasma.

We note that each operator can be represented by either of two different charges; this is a characteristic feature of the background charge construction. We have a certain freedom in how we pick either of these charges for each operator, and there is more than one equivalent representation of each conformal block. One particular construction is discussed in [13]. However, these details are not relevant at the moment. For our purposes, it is sufficient to know that, together with the background charge, our plasma is neutral, and that, roughly speaking, we need one screening charge for each singularity encountered in the wave function. There are $\frac{N}{2}$ singularities in the $c = \frac{1}{2}$ part of (4), making the total number of screening charges $\propto \frac{N}{2}$, a number which is large when the total number of the electrons N is large.

Therefore we arrive at the central point of this paper: the Pfaffian state can be mapped to a plasma *with three species of charges which interact via two different logarithmic interactions*, i.e.

$$|\Psi_{\text{Pf}}|^2 = \int \prod_A d^2 w_A e^{-\beta \Phi(z_i, \eta_\alpha, w_A)} \quad (9)$$

The inverse temperature, β , is 1, and Φ is the potential energy of a classical two-dimensional gas of logarithmically-interacting particles corresponding to:

- (a) electrons, which have charge $\sqrt{2}$ with respect to the first interaction ($c = 1$) and charge $3/\sqrt{6}$ or $-2/\sqrt{6}$ with respect to the second ($c = 1/2$)
- (b) quasiholes, which have charge $1/2\sqrt{2}$ with respect to the first interaction ($c = 1$) and charge $3/(2\sqrt{6})$ or $-1/2\sqrt{6}$ with respect to the second ($c = 1/2$)
- (c) the screening charges, which are neutral with respect to the first interaction ($c = 1$) and carry charge $4/\sqrt{6}$ or $-3/\sqrt{6}$ with respect to the other ($c = 1/2$)
- (d) the background charge, which is also neutral with respect to the first interaction ($c = 1$) and carry charge $-1/\sqrt{6}$ with respect to the second ($c = 1/2$).

The squared norm of the Pfaffian state is the partition function of this plasma. The squared modulus of the Pfaffian wavefunction (9) – which, naively, is a mess – is, in fact, the Boltzmann weight of this plasma with the screening charges already integrated out. The integration of the $N/2$ w_A 's does not affect the $c = 1$ interaction and, at least partially, screens the $c = 1/2$ interactions among the z_i 's and η_α 's. Hence, if we write (9) as the Boltzmann weight of some effective Hamiltonian, $\tilde{\Phi}$, it will have two species of particles which interact logarithmically through the $c = 1$ interaction and also through some weaker interaction (perhaps short-ranged) which is the partially screened $c = 1/2$ interaction. Though this latter interaction may be very complicated, we expect that (9) still essentially describes a plasma even after the w_A 's have been integrated out.

Quantum Liquid Nature and Electron Density Matrix. This observation allows us to immediately compute various properties of this state. Following Laughlin, we anticipate that it is a liquid because the associated plasma is above the crystallization temperature. As a result, we can compute its electron density matrix

$$\rho(z, z') = \int \prod_{i=2}^N d^2 z_i \Psi_{Pf}^*(z', z_2, \dots) \Psi_{Pf}(z, z_2, \dots) \quad (10)$$

We note that $\rho(z, z')$ is given by a partition function of our plasma with one charge held fixed at z ; in a liquid, it must be independent of z . On the other hand,

$$\rho(z, z') = \exp\left(-\frac{1}{4}(|z|^2 + |z'|^2)\right) G(z, \bar{z}') \quad (11)$$

where $G(z, \bar{z}')$ is analytic in its arguments. The only way to satisfy these two constraints is if

$$\rho(z, z') = \exp\left(-\frac{1}{4}(|z|^2 + |z'|^2) + \frac{1}{2}z\bar{z}'\right) \quad (12)$$

just as in the Laughlin states. There is no sign of pairing or any delicate power-law correlations in the electron Green function. Indeed, we don't expect such correlations in multi-electron Green functions either; all of these should be Gaussian or exponentially localized as a result of the gap. The interesting correlations should appear in Green functions of the non-local order parameter [1]. These correlations are related to the 'topological ordering' of the state and, hence, should be more stable against local perturbations than putative subtle correlations in the electron Green functions would be. Unfortunately the plasma analogy in the form presented here does not seem to allow to compute them.

Finally, we can argue, following Laughlin [8], that the charges of the fractionally charged particles are localized around their positions, due to the effective plasma repulsion. The quasiholes do not have power-law tails in their charge distribution as one might have naively guessed from the form of the wavefunctions and from the fact that they carry half of a flux quantum.

Fractional Charge and Abelian Statistics in the Two-Quasihole State. There is a unique two-quasihole state (see [4]) if the quasiholes are fixed at positions η_1 and η_2 . It equal to a conformal block with twist fields and appropriate $c = 1$ vertex operators inserted to represent the quasiholes [1,4]. According to our plasma analogy, the squared norm of the two-quasihole state is the partition function of our two-component plasma with two external charges (of magnitudes $1/2\sqrt{2}$ with respect to the $c = 1$ interaction and $3/(2\sqrt{6})$ or $-1/2\sqrt{6}$ with respect to the $c = 1/2$ interaction) held fixed at η_1 and η_2 . The plasma will screen these charges, so the partition function will be independent of η_1 and η_2 when their separation is larger than the screening length:

$$\left\langle \Psi_{Pf}^{2\text{-holes}}(\eta_1, \eta_2) | \Psi_{Pf}^{2\text{-holes}}(\eta_1, \eta_2) \right\rangle = Z_{\text{two comp. plasma}}(\eta_1, \eta_2) = \text{const.} \quad (13)$$

and consequently the non-trivial part of the Berry connection of these wave functions vanishes (the trivial part, coming from the Gaussian, gives the fractional charge of the quasiholes, $1/4$)

$$\begin{aligned} & \left\langle \Psi_{Pf}^{2\text{-holes}}(\eta_1, \eta_2) \left| \frac{\partial}{\partial \eta_1} \right| \Psi_{Pf}^{2\text{-holes}}(\eta_1, \eta_2) \right\rangle - \text{term coming from the Gaussian} = \\ & \frac{\partial}{\partial \eta_1} \left\langle \Psi_{Pf}^{2\text{-holes}}(\eta_1, \eta_2) | \Psi_{Pf}^{2\text{-holes}}(\eta_1, \eta_2) \right\rangle = 0 \end{aligned} \quad (14)$$

Therefore we can read the fractional statistics of the quasiholes of the Pfaffian state from the monodromy of the conformal block which, in this case, is determined by a factor of $(\eta_1 - \eta_2)^\alpha$. Interestingly, $\alpha = 0$ or, in other words, in a two-quasihole state, the two quasiholes are bosonic with respect to each other [1,4]. The fractional statistics which is concomitant with the fractional charge is cancelled by the fractional statistics resulting from the Pfaffian part of the wavefunction. Since the charge and statistics of the quasiholes are different, it should actually be easier to experimentally disentangle them in the Pfaffian state than in a Laughlin state.

Non-Abelian Statistics of 2n-Quasihole States. Now let us take up the less trivial case of four quasiholes. There are two possible four-quasihole wave functions when the quasiholes are located at $\eta_1, \eta_2, \eta_3, \eta_4$. We call them Ψ_0 and $\Psi_{\frac{1}{2}}$, following [4]. According to (8), the plasma analogy allows us to compute only the combination

$$\langle \Psi_0 | \Psi_0 \rangle + \left\langle \Psi_{\frac{1}{2}} | \Psi_{\frac{1}{2}} \right\rangle = Z_{\text{two comp. plasma}}(\eta_1, \eta_2, \eta_3, \eta_4) = \text{const.} \quad (15)$$

This is not enough to argue that the braiding statistics of the quasiholes can be read off the conformal blocks. What we need is the somewhat stronger statement,

$$\langle \Psi_i | \Psi_j \rangle = C_{ij}, \quad i, j = 0, \frac{1}{2} \quad (16)$$

where C_{ij} is some constant (independent of the positions of quasiholes) matrix, which, if true, implies that the nonabelian Berry connection vanishes

$$A_{ij} = \left\langle \Psi_i \left| \frac{\partial}{\partial \eta} \right| \Psi_j \right\rangle = \frac{\partial}{\partial \eta} C_{ij} = 0 \quad (17)$$

To derive (16) we apparently need to generalize (8). However, we can instead look at the problem in the following way. (8) represents a single valued function, understood as a function of η_i and $\bar{\eta}_i = \eta_i^*$. However, as a function of η_i only, with $\bar{\eta}_i$ viewed formally as an independent variable, (8) is not single-valued. Moreover, by taking one η_i around another (while keeping $\bar{\eta}_i$ at bay) we can transform (8) into an arbitrary combination of conformal blocks. For example, if we take η_1 around η_2 while holding $\bar{\eta}_1$ and $\bar{\eta}_2$ as well as $\eta_3, \bar{\eta}_3, \eta_4, \bar{\eta}_4$ fixed, then $\Psi_0 \rightarrow \Psi_0$ while $\Psi_{\frac{1}{2}} \rightarrow -\Psi_{\frac{1}{2}}$. If

we instead take η_1 around η_4 , while holding the rest fixed, $\Psi_0 \rightarrow \Psi_{\frac{1}{2}}$ and $\Psi_{\frac{1}{2}} \rightarrow \Psi_0$. Hence, with these two braiding operations we can transform $\langle \Psi_0 | \Psi_0 \rangle + \langle \Psi_{\frac{1}{2}} | \Psi_{\frac{1}{2}} \rangle$ into $\langle \Psi_0 | \Psi_0 \rangle - \langle \Psi_{\frac{1}{2}} | \Psi_{\frac{1}{2}} \rangle$, and $\langle \Psi_0 | \Psi_{\frac{1}{2}} \rangle \pm \langle \Psi_{\frac{1}{2}} | \Psi_0 \rangle$, from which we can obtain the desired inner products (16). In other words, after performing these braiding operations, we can express the above combinations of inner products as analytic continuations of $Z_{\text{two comp. plasma}}(\eta_1, \eta_2, \eta_3, \eta_4)$. The crux of the matter is now: can these analytically continued partition functions also be interpreted as partition functions of plasmas, albeit with modified interactions? If so, then we can, as usual, invoke plasma screening and conclude that all of the inner products are independent of the positions of the quasiholes.

At the moment, we do not have a proof that this is so. However, the following line of reasoning strongly supports this contention. Earlier, we argued that integration of the w_A 's should result in some sort of plasma with partially screened interactions. If we then proceed and integrate out the z_i 's, we expect total screening of the interactions between the η_α 's and, hence, the result (15). Suppose, instead, that before integrating the z_i 's we first carry out one of the analytic continuations described above. This will not effect the $c = 1$ interaction since the area enclosed by the path of the η_α 's can be made arbitrarily small and the $c = 1$ interaction will only be sensitive to the analytic continuation if one of the z_i 's crosses this path. The $c = 1/2$ interaction can be affected, but only through the introduction of phases which can change as z_i 's approach η_α 's. If these phases interfere destructively, the off-diagonal matrix elements vanish and the diagonal matrix elements are equal and constant. At worst, they can interfere constructively, in which case the other matrix elements are in the same situation as (15). This scenario, too, leads to the desired result (16) if we again appeal to plasma screening.

If this argument proves to be correct, we can conclude (by straightforwardly extending it to the $2n$ -quasihole case) that the statistics of quasiholes in the Pfaffian state is given by spinor representations of $SO(2n)$, as was conjectured in [4].

Discussion. As we discussed in the introduction, some quantum Hall wavefunctions can be reproduced by conformal field theories, and it is natural to conjecture that the braiding properties of the former are given by those of the latter [1,2]. In this paper, we have attempted to answer the question ‘why?’ The basic reason, we submit, is that these states are all related – through the background charge construction [12] – to Coulomb gases. As a result of plasma screening in these Coulomb gases, wavefunctions given by conformal blocks have trivial Berry connections; their braiding properties are determined entirely by the monodromies of the conformal blocks.

Thus far, we have focussed on the Pfaffian state. However, it is clear from our discussion that the arguments we have used are quite general and may be applied to any quantum Hall state with wavefunctions given by the conformal blocks of a theory with a background charge construction. To the casual observer, the background charge representation of a conformal block may seem like a complicated integral representation. However, it is perfectly tailored for the present context because the electron coordinates must also be integrated in the calculation of Berry matrices. Consequently, the screening charges are on the same footing as the electrons and the representation (9) is a natural generalization of the Coulomb gas representation of the Laughlin states.

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